Directed hyper-graphs and dynamical systems

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COMPLEX NETWORKS

Directed hyper-graphs

The building blocks of a directed hyper-graph are:

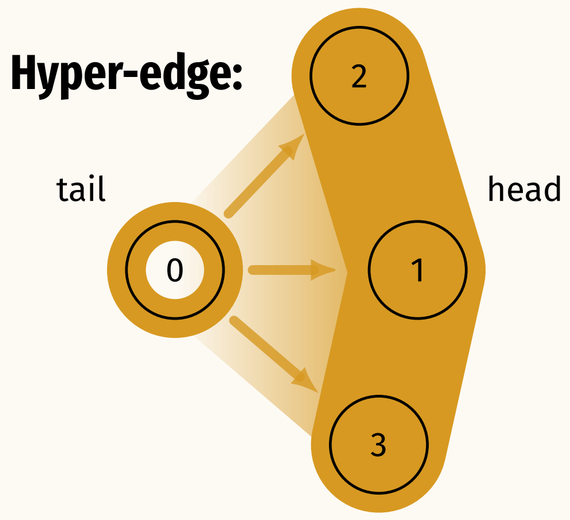
nodes a set of items interacting through hyper-edges;

hyper-edges a set of *tail-head* pairs;

tail the hyper-edge entry point (usually a node);

head the hyper-edge exit points (a set of nodes).

Tail and *head* assure directionality, see the Figure.



node	role
0	tail
1, 2, 3	head

Dynamics on hyper-graphs Consider a walker visiting the nodes, or a Markov-chain:

- At time t the walker is on node i;
- the walker selects a hyper-edge whose tail is attached to i, with probability that depends on the size of the head;
- at time $t+1$ the walker transition to one of the nodes of the corresponding head.

Transition probability:

$$p_{i\to j} \propto \sum_{e\in E_{ij}} |e|^{\tau}$$

where E_{ij} is the set of hyper-edges for which node i belong to the tail and node j to the head, and $|e|$ is the size of the head. In particular:

$\tau = -1 \to$ All tails are selected with the same probability

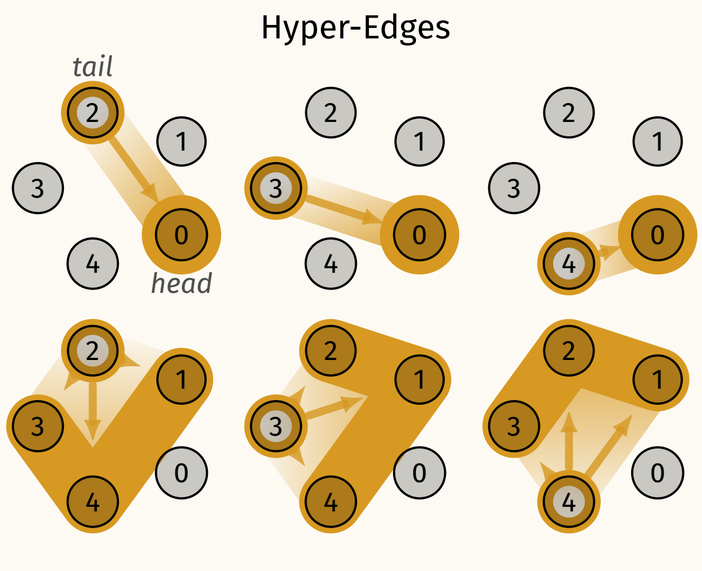
$\tau = 0 \to$ Tails are selected with a probability proportional to the size of the head.

TL,DR

- We introduce a directed hyper-graph. The directed hyper-graph has hyper-edges which **embed directionality**.
- The dynamics depend on the parameter τ which can favour or penalize traversing hyper-edges with larger heads.
- The dynamics on the hyper-graph can be described by a random walk on a suitable **effective adjacency matrix**. Such matrix defines an exact correspondence between the dynamics on the hyper-graph and those on the effective graph (with pairwise interactions).
- Measures and analysis of the dynamical system are applicable to both the hyper-graph and the corresponding effective graph.

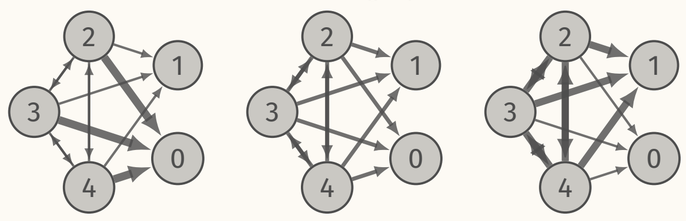
Ranking on hyper-graphs

Dependence of page-rank on the dynamical parameter τ:



$\tau=-1$		$\tau=0$		$\tau=1$	
ranking	prob	ranking	prob	ranking	prob
0	0.325	1	0.224	1	0.251
1	0.186	0	0.224	2	0.2
2	0.163	2	0.184	3	0.2
3	0.163	3	0.184	4	0.2
4	0.163	4	0.184	0	0.149

Effective graphs



The importance of nodes **0** and **1** depends on the value of the dynamical parameter τ. The parameter τ governs the dynamics and changing its value can invert the node rank.

Twitter and Covid-19

A real world example from the discussion on vaccines and Covid-19 on the French-speaking Twitter.
Hyper-edges corresponds to **tweets** and **retweets** such that the information flows from tweeting users to retweeting users:

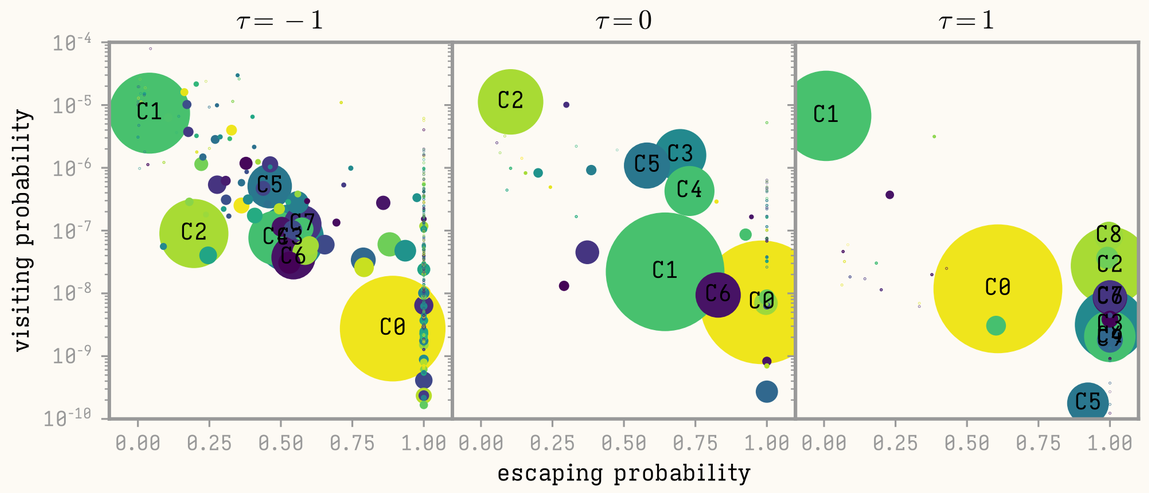
tweeting user	$\to$ *tail*
retweeters	$\to$ *head*

Communities that maximize auto-covariance on the hyper-graph, maximize modularity in the effective pairwise graph.

$\tau = -1$ emphasizing tweets, community structures are more heterogeneous.

$\tau = 0$ emphasizing retweets, a higher fraction of vaccine-critical information reaches outer communities.

This means that retweets from highly retweeted tweets have more chance to reach users on a different community.



Community structure on the vaccine discussion on Twitter. Communities are those that maximize the auto-covariance of the information flow. The plots show the probability that vaccine-critical content reaches outer communities (escaping probability) and the amount of vaccine-critical information flowing through each node of the community ([node] visiting probability).
At higher values of τ:

less communities smaller communities merge into the larger ones;

higer escape probability between-community retweet probability increases.